A canonical stratification of the moduli of isolated hypersurface singularities

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This note is a complement to the paper [Bj-La], and the monograph [La-Pf], on which it depends notationally.
If $f:=\mathrm{Spf}(k[[\underline{x}]]/(f))$ where $f\in k[\underline{x}]=k[x_1,\ldots,x_n]$ is an isolated hypersurface singularity, we associate to f a graded Lie algebra $L^*(f)= L^0(f)\oplus L^1(f)$, where $L^0(f):=\mathrm{Der}(k[[\underline{x}]]/(f))/\mathrm{Der}_\pi$, Der_π being the Lie ideal generated by the trivial derivations of the form, $E_{ij}\in \mathrm{Der}(k[[\underline{x}]]/(f))$, $E_{ij}(x_k)=0$ for $k\neq i,j$, $E_{ij}(x_i)= \partial f/\partial x_j$, and $E_{ij}(x_j)=-\partial f/\partial x_i$, and where $L^1(f):=(\underline{x})k[[\underline{x}]]/(f,(\underline{x})(\partial f/\partial x_i)_i)$,the André cohomology of the singularity, is a representation of $L^0(f)$, see [La-Pf], § 4.
It is easy to see that $\dim_k L^0(f)=\dim_k k[[\underline{x}]]/(f,\partial f/\partial x_i)_i= \tau(f)$. the Tjurina number of the singularity. Put $\tau_*(f) = \dim_k L^1(f)$.
Associating to f the graded Lie algebra $L^*(f)$, defines a map $\mathbf{M}_\tau\to \mathbf{L}_d$ from the moduli space of hypersurface singularities of constant Tjurina number $\tau(f)=\tau$,to the set of isomorphism classes of Lie algebras of dimension, $d=\dim_k L^*(f)$.
Unfortunately, $\mathbf{L}_d$ is not, in general, equipped with a scheme structure, nor with a natural structure of an algebraic space.
However, let $\mathbf{Lie}_d$ be the scheme of all Lie algebra laws of dimension d, and $\mathbf{K}^n$,the corresponding universal Lie algebra defined on $\mathbf{Lie}_d$. Then we proved, in [Bj-La], that there exists, in the category of algebraic spaces, a good quotient $\mathbf{L}(\underline{h})$, $\underline{h}=(h_0,\ldots,h_d)$, of the subspace,

$$\mathbf{Lie}_d(\underline{h}):=\{t\in \mathbf{Lie}_d \mid \dim_k H^i(\mathbf{K}^n(t),\mathbf{K}^n(t))=h_i, i=0,..,d\}$$

by the action of $Gl_n(k)$. Moreover, the restriction of the map $\mathbf{M}_\tau\to \mathbf{L}_d$ to the inverse image $\mathbf{M}(\underline{h})$ of $\mathbf{L}(\underline{h})$ defines a morphism.

The restriction of this morphism to a neighborhood of f, i.e. to the modular substratum $\mathbf{H}_0(f)$ (the prorepresenting substratum of [La-Pf]), is defined by a flat family of H_0-Lie algebras $\Lambda^*(f)= \Lambda^0(f) \oplus\Lambda^1(f)$ constructed in the following way: Let F be the versal family of f defined on H, and F_0 the restriction of F to H_0. Consider the map $Der_H(F)\to Der_{H_0}(F_0)$ and let $\Lambda^0(f)$ be the cokernel. It is a flat H_0-module, and has a natural H_0-Lie algebra structure. Moreover it acts on the first cohomology group for the singularity F_0 defined on H_0, i.e. on

$$H^1{}_*(F_0):=(\underline{x})H_0[\underline{x}]/(F_0,(\underline{x})(\partial F_0/\partial x_i)).$$

Put

$$\Lambda^1(f)=H^1{}_*(F_0)$$

$$\mathbf{H}(\underline{h}):=\{t\in \mathbf{H_0}|\ rk_t H^i(H_0,\Lambda^*,\Lambda^*)=h_i,\ i=0,\dots,d\},$$

and observe that the fiber $\Lambda^*(t)$ of the family Λ^* at the point t is the Lie algebra $L^*(F(t))$. Now the restriction of Λ^* to $\mathbf{H}(\underline{h})$ defines a morphism of algebraic spaces

$$l(\underline{h}): \mathbf{H}(\underline{h}) \to \mathbf{L}(\underline{h}).$$

The main result of [Bj-La] is the following,

Theorem Let $f(x,y) = x_1{}^n+x_2{}^n$. Then there exists an open neighborhood $\mathbf{U}$ of $\underline{0}$ in $\mathbf{H}_0(f)$, such that for every $\underline{h}\in \mathbf{Z}^{d+1}$, the restriction of the morphism of algebraic spaces

$$l(h): \mathbf{H}(\underline{h}) \to \mathbf{L}(\underline{h})$$

to $\mathbf{U}\cap\mathbf{H}(\underline{h})$, is an immersion.

In fact we prove a slightly stronger result, see (6,0), loc.cit., but the above version will be sufficient to illustrate the theme of this note. Anyway we have reasons to believe that the following should hold,

Conjecture Let f be any isolated hypersurface singularity, and let $\underline{h}\in \mathbf{Z}^{d+1}$, then the morphism

$$l^*: \mathbf{M}(\underline{h}) \to \mathbf{L}(\underline{h})$$

is an immersion.

Notice that in contrast to $\mathbf{L}(\underline{h})$, which is a fine moduli, the modular stratum $\mathbf{H}_0$ is not even a coarse moduli space. There is a, usually nontrivial, discrete equivalence relation $\mathbf{r}$ on $\mathbf{H}_0$, identifying points corresponding to isomorphic fibers (see [La-M-Pf], p.274). The Theorem above therefore shows that the stratification $\{\mathbf{H}(\underline{h})\}_{\underline{h}}$ of $\mathbf{H}_0$ is finer than the stratification defined by the discriminant of **r**.

Definition The stratification $\{\mathbf{H}_0(f)(\underline{h})\}_{\underline{h}\in \mathbf{Z}^{d+1}}$ of the modular stratum $\mathbf{H}_0 := \mathbf{H}_0(f)$ of an isolated hypersurface singularity f will from now on be called the canonical stratification of the modular substratum of f.

We know, by examples, that the canonical stratification , is highly nontrivial, certainly for quasihomogeneous hypersurfaces (as we have shown that it is finer than the discriminant filtration of the discrete equivalence relation on $\mathbf{H}_0$, see above, and [La-M-Pf]), but also for generic μ-constant deformations of such hypersurfaces, see the example of the family $f(t,u)= x_1{}^5+x_2{}^{11}+tx_1{}^2x_2{}^7+2ux_1{}^4x_2{}^2+u2x_1{}^3x_2{}^4$, of [La-Pf], § 5., along which the embedding dimension of the modular substratum, and also μ, changes. However, as we shall show, it has some nice general properties.

Definition. Let **C** be a subscheme of **X**. We shall say that **C** is confined to **X**, if the forgetful morphism $\mathrm{Hilb}_{\mathbf{X}\supseteq \mathbf{C}} \to \mathrm{Def}_{\mathbf{C}}$ is onto.

One may easily convince oneself about the truth of the following assertions :

(1):If **C** is irreducible, and X is reduced, then **C** is confined to **X** if and only if **C** is rigid.

(2):If **X** is a double hypersurface, and **C** is its reduced subscheme, then **C** is confined to X.

(3):There exists a non reduced **X** such that $\mathbf{C}=\mathbf{X}_{red}$ is not confined to **X.**

The only excuse I have for proposing the above definition is that it makes it easier to state, presicely, the main, and only, result of this Note.

But first, some notations. Let f be any isolated hypersurface singularity. Put $\mathbf{H}_0(\underline{h}) := \mathbf{H}_0(f)(\underline{h})$, and let $\Lambda^* := \Lambda^0(f) \oplus \Lambda^1(f)$, be the universal family of graded Lie algebras on $\mathbf{H}_0$. When there is no danger of confusion, we shall just denote by $\Lambda^* := \Lambda^0 \oplus \Lambda^1$ the restriction $\Lambda^*(\underline{h}) := \Lambda^0(\underline{h}) \oplus \Lambda^1(\underline{h})$ of $\Lambda^0(f) \oplus \Lambda^1(f)$ to $\mathbf{H}_0(\underline{h})$, or to any other subscheme of $\mathbf{H}_0(\underline{h})$.

Proposition. (i) There is a canonical k-linear injective map

$$o: H^1(k, \mathbf{H}_0, \mathbf{H}_0) \to H^1(O_{\mathbf{H}_0}, \Lambda^0, \Lambda^1).$$

(ii) $\mathbf{H}_0$ has constant embedding dimension along any $\mathbf{H}_0(f)(\underline{h})$, $h \in Z^{2\tau.+n-1}$

(iii) Let $O := O_{\mathbf{H}_0(\underline{h}),t}$ be the the completion of the local ring of $\mathbf{H}_0(\underline{h})$ at t, then there is a canonical k-linear injective map

$$o(\underline{h}): H^1{}_*(k, O, O) \to H^1(O, \Lambda^0, \Lambda^1)$$

(iv). Any irreducible component of O is, as a singularity, confined to O.

Proof. The subscheme $\mathbf{H}_0(\underline{h})$ is the universal subscheme of $\mathbf{H}_0$ on which the $O_{\mathbf{H}_0}$-module $H^i(O_{\mathbf{H}_0(\underline{h})}, \Lambda^*, \Lambda^*)$ is flat of rank h_i, i =1,2,... 2τ -1+n. This means that if we are given a morphism of schemes $\pi: \underline{S} \to \underline{H}$ such that $p^*(\Lambda^*)$ is an $O_{\underline{S}}$-(flat) Lie algebra with $O_{\underline{S}}$-flat cohomology of rank h_i, then $\pi: \underline{S} \to \underline{H}$ factors through $\mathbf{H}_0(\underline{h})$.
I claim there is a natural linear map

$$o: H^1(k, \mathbf{H}_0, \mathbf{H}_0) \to H^1(O_{\mathbf{H}_0}, \Lambda^0, \Lambda^1)$$

and, similarly, for any $\underline{h}$ a natural map

$$o(\underline{h}): H^1{}_*(k, \mathbf{H}_0(\underline{h}), \mathbf{H}_0(\underline{h})) \to H^1(O_{\mathbf{H}_0(\underline{h})}, \Lambda^0(\underline{h}), \Lambda^1(\underline{h})).$$

The construction being the same in the two cases, lets construct $o(\underline{h})$. An element $\xi \in H^1{}_*(k, \mathbf{H}_0(\underline{h}), \mathbf{H}_0(\underline{h}))$ correspond to a lifting of $\mathbf{H}_0(\underline{h})$ to $k[\varepsilon]$, say $\mathbf{H}_0(\underline{h})_\xi$. A derivation $\delta \in \mathrm{Der}_{O_{\mathbf{H}_0(\underline{h})}}(F(\underline{h}))$ corresponds to an automorphism $id+\delta\eta$ of $F(\underline{h}) \otimes k[\eta]$ defined on $\mathbf{H}_0(\underline{h}) \otimes k[\eta]$, extending the identity of $F(\underline{h})$ on $\mathbf{H}_0(\underline{h})$. Since $F(\underline{h})$ is a hypersurface, there exists a lifting $F(\underline{h})_\xi \otimes k[\eta]$ of $F(\underline{h}) \otimes k[\eta]$ to $\mathbf{H}_0(\underline{h})_\xi \otimes k[\eta]$. The obstruction $o'(\xi, \delta)$ for lifting $id+\delta\eta$ to $F(\underline{h})_\xi \otimes k[\eta]$ sits in $\Lambda^1(\underline{h}) = H^1{}_*(O_{\mathbf{H}_0(\underline{h})}, F(\underline{h}), F(\underline{h}))$ (multiplied by $\varepsilon\eta$). Using the general facts on obstructions of compositions of morphisms, see [La1],chap.2., one easily proves that the map associating $o'(\xi, \delta)$ to δ is a derivation,(pick two infinitesimals η_1, η_2 and consider $\sigma_i = id+\delta_i\eta_i$, i=1,2, and compute the obstruction for lifting the automorphism $\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}$). Therefore $o'(\xi)$ is an element of $\mathrm{Der}(\mathrm{Der}_{O_{H_0(\underline{h})}}(F(\underline{h})),\Lambda^1(\underline{h}))$. However, $o'(\xi)$ is immediately seen to be zero on the image of $\mathrm{Der}_{O_H}(F)$ in $\mathrm{Der}_{O_{H_0(\underline{h})}}(F(\underline{h}))$, thereby inducing an element of $\mathrm{Der}(\Lambda^0(\underline{h}),\Lambda^1(\underline{h}))$. Moreover if I choose another lifting $F(\underline{h})_\xi$ of $F(\underline{h})$, to $\mathbf{H}_0(\underline{h})_\xi$, corresponding to the choice of an element λ of $\Lambda^1(\underline{h})= H^1{\cdot}(O_{\mathbf{H}_0(\underline{h})},F(\underline{h}),F(\underline{h}))$, then the difference in the resulting derivations , $o'(\xi)$, is the trivial derivation defined, presicely, by the element λ . The map $o(\underline{h})$, that we want, is defined by associating to ξ the class of $o'(\xi)$ in $H^1(O_{\mathbf{H}_0(\underline{h})}, \Lambda^0(\underline{h}),\Lambda^1(\underline{h}))$.

Suppose for some $\xi\in H^1(k,\mathbf{H}_0,\mathbf{H}_0)$, that $o(\xi)=0$. This means that $\mathrm{Der}_{O_{H_0}}(F_0)$, and therefore also Λ^*, lifts to $O_{H_0\xi}$ as $\mathrm{Der}_{O_{H_0\xi}}(F_{0\xi})$, for some lifting $F_{0\xi}$. But, since $\mathbf{H}_0$ is the universal subscheme of $\mathbf{H}$ to which $\mathrm{Der}_k(f)$ lifts, this means that the lifting $\mathbf{H}_{0\xi}$ of $\mathbf{H}_0$ splits, thus $\xi=0$. This proves point (i) of the Proposition.

Suppose, for some $\xi\in H^1{*}(k,O_{\mathbf{H}_0(\underline{h}),t}, O_{\mathbf{H}_0(\underline{h}),t})$, that $o(\underline{h})(\xi)=0$, then, as above the Lie algebra $\Lambda^*(\underline{h})$ lifts to $O_{\mathbf{H}_0(\underline{h}),t\,\xi}$, and we shall want to prove that then the lifting $O_{\mathbf{H}_0(\underline{h}),t\,\xi}$ of $O_{\mathbf{H}_0(\underline{h}),t}$ splits.

Remember that we are talking about deformations of singularities, in particular of pointed schemes, therefore there is a section of $F(\underline{h})_\xi$ on $\mathbf{H}_0(\underline{h})_\xi$ and since we are considering the complete local ring $O=O_{\mathbf{H}_0(\underline{h}),t}$ as a singularity, we also have given a section of $O_\xi=O_{\mathbf{H}_0(\underline{h}),t\,\xi}$ on $k[\varepsilon]$. This implies the existence of a diagram of exact sequences

$$
\begin{array}{ccc}
 & & 0 \\
\downarrow & & \downarrow \\
H^i(O,\Lambda^*,\Lambda^*)\varepsilon & \rightarrow & H^i(k,L^*(F(t)),L^*(F(t)))\varepsilon \\
i\downarrow & & \downarrow \\
H^i(O_\xi,\Lambda^*_\xi,\Lambda^*_\xi) & \rightarrow & H^i(k[\varepsilon],L^*(F(t))[\varepsilon],L^*(F(t))[\varepsilon]) \\
\downarrow & & \downarrow \\
H^i(O,\Lambda^*,\Lambda^*) & \rightarrow & H^i(k,L^*(F(t)),L^*(F(t))) \\
\downarrow & & \downarrow \\
 & & 0
\end{array}
$$

For $i=\dim_k L(f)^*=:d$, the left sequence ends with a surjection, and the right hand sequence is always split. Since the top and bottom modules are free of rank h_i and the top and bottom horizontal

maps are onto, the middle horizontal map is also onto. We shall use this to prove, by induction on the integer i, starting with the maximal one d, that the map **i**, in the upper right corner is injective.
This will prove that the $O' = O_{H_0(\underline{h}),t_\xi}$-modules

$H^i := H^i(O', \Lambda^*_\xi, \Lambda^*_\xi)$ are flat, and of rank h_i, thereby proving, in the same way as above, that $\xi = 0$.
We start by proving that $\mathrm{Tor}^{O'}_1(H^i, k[\varepsilon]) = 0$. By the surjectivity of the middle horizontal map above, we know that $\mathrm{Tor}^{O'}_0(H^i, k[\varepsilon]) = H^i(k[\varepsilon], L^*(F(t))[\varepsilon], L^*(F(t))[\varepsilon])$. Moreover, since Λ^*_ξ is O'-flat, there is an obvious spectral sequence given by

$$E_2^{-p,i+p} = \mathrm{Tor}^{O'}_p(H^{i+p}, k[\varepsilon])$$

converging to $H^i(k[\varepsilon], L^*(F(t))[\varepsilon], L^*(F(t))[\varepsilon])$. Since $H^{d+p}=0$ for $p \geq 1$, the differentials $d_2 : E_2^{-p,i+p} \to E_2^{2-p,i+p-1}$ entering and leaving $E_2^{-1,d} = \mathrm{Tor}^{O'}_1(H^d, k[\varepsilon])$ must be zero, so that

(1) $\qquad \mathrm{Tor}^{O'}_1(H^d, k[\varepsilon]) = 0,$

and in fact all $\mathrm{Tor}^{O'}_i(H^d, k[\varepsilon]) = 0$, $i = 1, 2, \ldots$ Now, considering the exact sequence of O'-modules $0 \to k\varepsilon \to k[\varepsilon] \to k \to 0$, and tensorizing with H^d, we have seen that we obtain an exact sequence, i.e. the right hand vertical split sequence of the diagram above. It follows that we also have,

(2) $\qquad \mathrm{Tor}^{O'}_1(H^d, k) = 0.$

(We could, of course have proved (2) first, and then (1),taking care of the fact that $k[\varepsilon]$ is not k^2 as an O'-module.) As usual, this implies that $\mathrm{Tor}^{O'}_1(H^d, M) = 0$, for any O'-module M of finite length, and therefore also for all O'-modules M of finite type, since any such is a projective limit of finite length modules. (Notice that the fact, that all finite length O'- modules are sucsesive extensions of k and or $k[\varepsilon]$, is a consequence of O' being a deformation of the local ring O, in the category of singularities.) But then H^d is O'-flat, and **i** is injective, such that, proceeding by induction, we prove that all H^i,are flat O'-modules, and in fact liftings of $H^i(O, \Lambda^*, \Lambda^*)$. This proves part (iii) of the Proposition.
The remaining assertions now follows from the fact that $H^i(O_{H_0(\underline{h})}, \Lambda^0, \Lambda^1)$, and $H^i(O, \Lambda^0, \Lambda^1)$ are, for every i, summands in $H^i(O_{H_0(\underline{h})}, \Lambda^*, \Lambda^*)$, and $H^i(O, \Lambda^*, \Lambda^*)$, respectively, and therefore flat

as $O_{H_0(\underline{h})}$, and O-modules, respectively. (ii) follows from the flatness of $H^0(O_{H_0(\underline{h})},\Lambda^0,\Lambda^1)$, since the tangent space of $\mathbf{H}_0$ at a point t is $H^0(k,L^0(F(t)),L^1(F(t)))$, and (iv) follows from the flatness of $H^1(O,\Lambda^0,\Lambda^1)$, coupled with the assumptions implying that for any irreducible component $O \to C$ of O, $H^1(k,C,C)$ is a torsion module, but $H^1(C,\Lambda^0,\Lambda^1)$ is not. **QED**

Corollary. If $\mathbf{H}_0(\underline{h})$ is reduced at the point t, then any irreducible component C of $O=O_{H_0(\underline{h}),t}$,is rigid, as a singularity.

Remark. I would not be surprised if it turned out that every $\mathbf{H}_0(\underline{h})$ is non-singular.

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